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Jacobi's last multiplier and symmetries for the Kepler problem plus a lineal story

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Abstract
We calculate the first integrals of the Kepler problem by the method of Jacobi's last multiplier using the symmetries for the equations of motion. Also we provide another example which shows that Jacobi's last multiplier together with Lie symmetries unveils many first integrals neither necessarily algebraic nor rational whereas other published methods may yield just one.

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1. Introduction

In his sixteenth lecture on dynamics [15] Jacobi uses his method of the last multiplier [12–14] to derive the components of the Laplace–Runge–Lenz vector for the two-dimensional Kepler problem. It is not surprising that Jacobi did not refer to the first integrals obtained as components of the Laplace–Runge–Lenz vector since Runge [35] and Lenz [25] had yet to write on the subject. However, one is surprised at the omission of a reference to the work of Laplace [19] by whom the components of the vector were derived fewer than 70 years earlier in somewhat less crowded times. Indeed one is a little surprised at Jacobi's neglect of the pioneering derivation by Ermanno [9], Herman [11] and Bernoulli [3] given Jacobi's prolonged exposures to Italian matters, among others his participation in the Congress of the Italian Scientists in 1843 at Lucca [2]. In his development Jacobi uses the approach to determine the last multiplier through a linear first-order partial differential equation.

A less classical approach to the derivation of the Laplace–Runge–Lenz vector is that of vectorial manipulation of the vector equation of motion by Bleuler and Kustaanheimo [5] and,

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in a more popular context, by Collinson [6, 7]. We recall that the reduced equation of motion for the Kepler problem is

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{1}$$

and that the angular momentum, $\mathbf{L} := \mathbf{r} \times \dot{\mathbf{r}}$, is a conserved quantity. The vector product of $\mathbf{L}$ with (1) is integrable to give the conserved vector

$$\mathbf{J} = \mathbf{L} \times \dot{\mathbf{r}} - \mu\hat{\mathbf{r}}. \tag{2}$$

This is, of course, the celebrated vector. In fact one can integrate (1) directly given the conservation of angular momentum to obtain Hamilton's vector [10], namely

$$\mathbf{K} = \dot{\mathbf{r}} + \mu\hat{\boldsymbol{\theta}}, \tag{3}$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the direction of increasing angle on the plane of the orbit.

The technique of vectorial manipulation of the equation of motion has been extended to a number of problems emanating from generalizations of the Kepler problem sharing a common property which is the possession of a conserved vector of a nature similar to that of the Laplace–Runge–Lenz vector [20–23].

When one considers the ease of integration of the Kepler problem, it is also a little surprising that the equation of motion, (1), possesses just five Lie point symmetries with the algebra $A_2 \oplus so(3)$. We refer to the easy manipulation of the equation of motion to obtain the first integrals rather than the formal notion of integrability which has a technical meaning not at the point of this remark. Although the Kepler problem has a sufficient number of point symmetries to guarantee reduction to a series of quadratures [27], integrability in the sense of Lie, the number of point symmetries is fewer than the order of the system and so one can only be surprised at the ease of derivation of the first integrals. These five symmetries are insufficient to specify completely (1). Krause [17, 18] remedied this deficit by an ingenious stratagem. This was to postulate the existence of a symmetry of the form

$$\Gamma = \left(\int \xi \, \mathrm{d}t\right) \partial_t + \eta_i \partial_{x_i}. \tag{4}$$

The nonlocality of the coefficient function of ∂_t did not intrude upon the determination of the form of (1) since the A_2 subalgebra contains ∂_t as one of its two elements—the other is the rescaling symmetry usually associated with the Laplace–Runge–Lenz vector and Kepler's third law—and so (1) is necessarily autonomous. With this device, Krause found a further three symmetries of the form

$$\boldsymbol{\Gamma} = \left(2\int \mathbf{r} \, \mathrm{d}t\right) \partial_t + r\mathbf{r}.\partial_{\mathbf{r}}, \tag{5}$$

where $\mathbf{r} = (x_1, x_2, x_3)$, and with these additional symmetries was able to specify completely (1) starting from the general equation

$$\ddot{\mathbf{r}} = \mathbf{f}(t, x_i, \dot{x}_i). \tag{6}$$

Subsequently Nucci [29] showed that the additional symmetries obtained by Krause had a point origin in a reduced system. The method of reduction of order [31] was shown to be applicable, *mutatis mutandis*, to a wide range of equations possessing a conserved vector of the type of the Laplace–Runge–Lenz vector [32]. When the Ermanno–Bernoulli constants were used to define the independent variables, (1) was shown [24] to be reducible to a two-dimensional simple harmonic oscillator plus a conservation law and, since the complete symmetry group of the two-dimensional simple harmonic oscillator is five dimensional [1], the number of symmetries required to specify completely the Kepler problem was smaller than originally reported by Krause. However, that number necessarily included the symmetries derived by

Krause so that the credit for spearheading a new development in a problem of over three and a quarter centuries existence remains his.

2. Jacobi's last multiplier

The method of Jacobi's last multiplier, M, provides a means to determine an integrating factor of the partial differential equation

$$Af = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} = 0 \tag{7}$$

or its equivalent associated Lagrange's system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \cdots = \frac{\mathrm{d}x_n}{a_n}. \tag{8}$$

The multiplier is given by

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = MAf, \tag{9}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = \det \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \omega_1}{\partial x_1} & & \dfrac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} = 0 \tag{10}$$

and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (7) or, equivalently, first integrals of (8). One can prove that each multiplier is a solution of a linear partial differential equation, namely

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0. \tag{11}$$

In general a different selection of integrals produces another multiplier, M'. An important property of the last multiplier is that the ratio, M/M', is a solution of (7), equally a first integral of (8). Naturally the ratio may be quite trivial.

In its original formulation the method of Jacobi's last multiplier required almost complete knowledge of the system, (7) or (8), under consideration. Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries was provided by Lie [26, pp 333–47]. A clear treatment of the formulation in terms of solutions/first integrals (pp 456–61) and symmetries (pp 461–4) is given by Bianchi [4]. If we know $n-1$ symmetries of (7)/(8), say

$$\Gamma_i = \xi_{ij}\partial_{x_j}, \qquad i = 1, n-1, \tag{12}$$

Jacobi's last multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}. \tag{13}$$

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. The differential equation to be solved for Jacobi's last multiplier is [15, p 126]

$$0 = \frac{\mathrm{d}\log(M)}{\mathrm{d}t} + \sum_{i=1}^{n} \frac{\partial W_i}{\partial w_i}, \tag{14}$$

where M is the multiplier and the equation of motion has components $\dot{w}_i = W_i$. Consequently, if each component of the vector field of the equation of motion is free of the variable associated with that component, i.e. $\partial W_i/\partial w_i = 0$ (no summation in this case), the last multiplier is a constant. This feature was recently put to good use with the Euler–Poinsot system [33].

3. Kepler problem

The original derivation of the Ermanno–Bernoulli constants was, naturally, in terms of Cartesian coordinates, but the facility of using polar coordinates has long been established for central force problems. Unfortunately in polar coordinates the vector field of the equation of motion does not have the attractive property that $\partial W_i/\partial w_i = 0$ which gives a first integral directly from the corollary. However, the Cartesian representation of the equations of motion has and so in this work we use the Cartesian form so that we can calculate the first integrals of the Kepler problem directly from the vector fields of the equation of motion and the symmetries. We should mention that Marcelli and Nucci [27] have shown that first integrals can be obtained by Lie group analysis even if the system under study does not come from a variational problem, i.e. we can find first integrals without making use of Noether's theorem [28]. The only limitation is the absence of one of the unknowns in the expression for the first integral. The first integrals correspond to the characteristic curves of determining equations of parabolic type which are constructed by the method of Lie group analysis, following the application of Nucci's method of reduction of order [29]. In [27] the method was also applied to the Kepler problem; using Lie symmetries the Cartesian components of the angular momentum were derived and the Kepler problem was reduced to a second-order linearizable equation. The Kepler problem is of the sixth order and autonomous so that one needs a 6×6 matrix which requires the equation of motion and five symmetries excluding ∂_t. Note that in general, i.e. for a nonautonomous system, it would be necessary to use a 7×7 matrix for a sixth-order system since there are seven variables. Effectively the autonomy of the system allows us to strike out a row and a column of the matrix.

We write the equation of motion as a six-dimensional system of first-order ordinary differential equations, namely

$$\begin{aligned}
&\dot{\omega}_1 = \omega_4 \qquad \dot{\omega}_2 = \omega_5 \qquad \dot{\omega}_3 = \omega_6 \\
&\dot{\omega}_4 = -\frac{\mu\omega_1}{\left(\omega_1^2 + \omega_2^2 + \omega_3^3\right)^{3/2}} \qquad \dot{\omega}_5 = -\frac{\mu\omega_2}{\left(\omega_1^2 + \omega_2^2 + \omega_3^3\right)^{3/2}} \\
&\dot{\omega}_6 = -\frac{\mu\omega_3}{\left(\omega_1^2 + \omega_2^2 + \omega_3^3\right)^{3/2}}.
\end{aligned} \tag{15}$$

The standard Lie point symmetries are

$$\begin{aligned}
\Gamma_1 &= \partial_t \\
\Gamma_2 &= 3t\partial_t + 2\omega_1\partial_{\omega_1} + 2\omega_2\partial_{\omega_2} + 2\omega_3\partial_{\omega_3} - \omega_1\partial_{\omega_1} - \omega_2\partial_{\omega_2} - \omega_3\partial_{\omega_3} \\
\Gamma_3 &= -\omega_3\partial_{\omega_2} + \omega_2\partial_{\omega_3} - \omega_6\partial_{\omega_5} + \omega_5\partial_{\omega_6}
\end{aligned}$$

$$\begin{aligned}\Gamma_4 &= \omega_3\partial_{\omega_1} - \omega_1\partial_{\omega_3} + \omega_6\partial_{\omega_4} - \omega_4\partial_{\omega_6}\\ \Gamma_5 &= -\omega_2\partial_{\omega_1} + \omega_1\partial_{\omega_2} - \omega_5\partial_{\omega_4} + \omega_4\partial_{\omega_5}\end{aligned} \tag{16}$$

and the three nonlocal symmetries of Krause are

$$\begin{aligned}\Gamma_6 &= \left(2\int \omega_1\,\mathrm{d}t\right)\partial_t + \omega_1\left(\omega_1\partial_{\omega_1} + \omega_2\partial_{\omega_2} + \omega_3\partial_{\omega_3}\right)\\ &\qquad + (\omega_2\omega_4 - \omega_1\omega_5)\,\partial_{\omega_5} + (\omega_3\omega_4 - \omega_1\omega_6)\,\partial_{\omega_6}\\ \Gamma_7 &= \left(2\int \omega_2\,\mathrm{d}t\right)\partial_t + \omega_2\left(\omega_1\partial_{\omega_1} + \omega_2\partial_{\omega_2} + \omega_3\partial_{\omega_3}\right)\\ &\qquad + (\omega_1\omega_5 - \omega_2\omega_4)\,\partial_{\omega_4} + (\omega_3\omega_5 - \omega_2\omega_6)\,\partial_{\omega_6}\\ \Gamma_8 &= \left(2\int \omega_3\,\mathrm{d}t\right)\partial_t + \omega_3\left(\omega_1\partial_{\omega_1} + \omega_2\partial_{\omega_2} + \omega_3\partial_{\omega_3}\right)\\ &\qquad + (\omega_1\omega_6 - \omega_3\omega_4)\,\partial_{\omega_4} + (\omega_2\omega_6 - \omega_3\omega_5)\,\partial_{\omega_5}.\end{aligned} \tag{17}$$

Without additional symmetries such as those provided by Krause, the calculation of the integrals using the determinental method of Jacobi's last multiplier would not be possible no matter which coordinate system one preferred. Andriopoulos *et al* [24] provided several symmetries additional to those derived by Krause. However, these are nonlocal in the coefficient functions of the dependent variables and so the nonlocality is not automatically removed by the symmetry of time translation, ∂_t. We note that Jacobi [15] used both Cartesian (Vorlesung 16) and polar (Vorlesung 24) coordinates in his treatment of the Kepler problem.

To construct the matrix for the calculation of the first integral, we use the vector field of (15) and five of the seven symmetries listed in (16) less Γ_1 and (17). The effect of the excision of the first column which belongs to the independent variable is to avoid inclusion of the nonlocal part of any of the nonlocal symmetries in (17) contained in the matrix. By way of illustration the matrix we obtain with the choice of $\Gamma_2 - \Gamma_6$ is

$$A_{78} = \begin{bmatrix} \omega_4 & \omega_5 & \omega_6 & -\dfrac{\mu\omega_1}{r} & -\dfrac{\mu\omega_2}{r} & -\dfrac{\mu\omega_3}{r}\\ 2\omega_1 & 2\omega_2 & 2\omega_3 & -\omega_4 & -\omega_5 & -\omega_6\\ 0 & -\omega_3 & \omega_2 & 0 & -\omega_6 & \omega_5\\ \omega_3 & 0 & -\omega_1 & \omega_6 & 0 & -\omega_4\\ -\omega_2 & \omega_1 & 0 & -\omega_5 & \omega_4 & 0\\ \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 & 0 & \omega_2\omega_4 - \omega_1\omega_5 & \omega_3\omega_4 - \omega_1\omega_6 \end{bmatrix}, \tag{18}$$

where we have used $r^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$ to keep the matrix as compact as possible, in which we have used the symmetries omitted in A_{ij} to identify the particular matrix. Using Maple 7 the evaluation of the determinant of A_{78} gives

$$\begin{aligned} D_{78} &= \left[\omega_1^2\omega_5^2 + \omega_1^2\omega_6^2 - 2\omega_4\omega_1\omega_2\omega_5 - 2\omega_4\omega_3\omega_6\omega_1 + \omega_4^2\omega_2^2\right.\\ &\qquad \left. + \omega_2^2\omega_6^2 - 2\omega_6\omega_2\omega_3\omega_5 + \omega_3^2\omega_5^2 + \omega_3^2\omega_4^4\right]\\ &\qquad \times\left[r^3\omega_1\omega_5^2 - \omega_4 r^3\omega_2\omega_5 - \omega_3^3\mu + \omega_6^2 r^3\omega_1 - \omega_1\mu\omega_3^2 - \omega_1\omega_2^2\mu - \omega_4 r^3\omega_3\omega_6\right]/r^3 \end{aligned} \tag{19}$$

(in actual fact for the machine computation the expansion for r in terms of the ωs was used) which we may write in the clearer form[2]

$$D_{78} = L^2 \left[\omega_1 \left(\omega_5^2 + \omega_6^2 - \frac{\mu}{r}\right) - \omega_4 \left(\omega_2\omega_5 + \omega_3\omega_6\right)\right], \tag{20}$$

where

$$L^2 := (\omega_2\omega_6 - \omega_3\omega_5)^2 + (\omega_3\omega_4 - \omega_1\omega_6)^2 + (\omega_2\omega_4 - \omega_1\omega_5)^2 \tag{21}$$

is the square of the magnitude of the angular momentum.

In a similar fashion we find

$$D_{68} = L^2 \left[\omega_2 \left(\omega_4^2 + \omega_6^2 - \frac{\mu}{r}\right) - \omega_5 \left(\omega_1\omega_4 + \omega_3\omega_6\right)\right] \tag{22}$$

and

$$D_{67} = L^2 \left[\omega_3 \left(\omega_4^2 + \omega_5^2 - \frac{\mu}{r}\right) - \omega_6 \left(\omega_1\omega_4 + \omega_2\omega_5\right)\right]. \tag{23}$$

In D_{78}, D_{68} and D_{67} we recognize the three Cartesian components of the Laplace–Runge–Lenz vector multiplied by the square of the angular momentum. However, if we were unaware that the angular momentum was conserved, we would have to use as first integrals the expressions given.

Immediately one wonders what results are found if different combinations of the symmetries are chosen for the rows of the matrix used to determine the first integral. We find the following. In each case the unmentioned row is the vector field of the equation of motion.

(i) If two of the nonlocal symmetries and the three rotation symmetries are used, i.e. a matrix of type A_{26} (aeq A_{27} or A_{28}), the determinant is zero.

(ii) If two of the nonlocal symmetries, two of the rotation symmetries and the rescaling symmetry are used, we find results of the type

$$D_{58} = L^2 L_5^2 \qquad D_{48} = L^2 L_5 L_4 \qquad D_{38} = L^2 L_5 L_3, \tag{24}$$

where $L_3 = \omega_2\omega_6 - \omega_3\omega_5$, $L_4 = \omega_3\omega_4 - \omega_1\omega_6$ and $L_5 = \omega_1\omega_5 - \omega_2\omega_4$ are the three components of the angular momentum, indicating the general result

$$D_{ij+3} = L^2 L_j L_i, \qquad i, j = 3, 4, 5. \tag{25}$$

(iii) The three of the nonlocal symmetries and two of the rotation symmetries are used,

$$D_{2i} = 0, \qquad i = 3, 4, 5. \tag{26}$$

(iv) If three of the nonlocal symmetries, one of the rotation symmetries and the rescaling symmetry are used,

$$D_{ij} = 0, \qquad i \neq j = 3, 4, 5. \tag{27}$$

The number of zeros may seem disappointing, but from previous experience it is not an unexpected outcome [33].

[2] As a practical tool, using the method of Jacobi's last multiplier has been greatly enhanced with the advent of symbolic manipulation codes which remove the multiplicitive drudgery from the computation. It is unfortunate that the result, as delivered by these codes, is not always particularly transparent.

4. First integrals galore

In [8] a method was proposed to solve second-order ordinary differential equations as an 'attempt to address algorithmically their solution', as the authors stated in the abstract. The authors presented four equations: they determined a first integral of each equation by using their procedure. In [30] it is shown that all four equations possess enough Lie point symmetries to make them integrable by quadrature if not linearizable. Moreover, the method of Jacobi's last multiplier as exemplified above for the Kepler problem is put at work in order easily to construct first integrals, a galore of them if the equation admits an eight-dimensional Lie symmetry algebra. In order to fulfil the requirement of anonymous referees we present here one of those examples as treated in [30].

In [8] the authors introduced the following equation of second order the first integral of which 'is not a rational function' (example 4)[3]:

$$yuu'' + uu' + 2yuu'^2 + yu'^2 = 0. \tag{28}$$

Using their proposed method Duarte *et al* derive the following nonrational first integral:

$$I = y + \tfrac{1}{2}\log(yuu'). \tag{29}$$

In [30] it is shown that equation (28) admits an eight-dimensional Lie symmetry algebra generated by the following eight operators:

$$\begin{aligned}
X_1 &= \tfrac{1}{4}\,\mathrm{e}^{2u}(2u-1)y\partial_y \\
X_2 &= \frac{1}{16u}\,\mathrm{e}^{2u}(2u-1)[4\log(y)yu\partial_y + (2u-1)\partial_u] \\
X_3 &= \frac{1}{u}(2u-1)\partial_u \\
X_4 &= \frac{1}{u}\log(y)[4\log(y)yu\partial_y + (2u-1)\partial_u] \\
X_5 &= \frac{1}{u}\,\mathrm{e}^{-2u}\partial_u \\
X_6 &= \frac{1}{u}\log(y)\,\mathrm{e}^{-2u}\partial_u \\
X_7 &= y\partial_y \\
X_8 &= \log(y)y\partial_y
\end{aligned} \tag{30}$$

which means that equation (28) is linearizable by means of a point transformation [26]. In [30] it is also shown how to find the linearizing transformation and thus the general solution of equation (28) is easily derived. Moreover the method of Jacobi's last multiplier is used with the purpose of finding first integrals.

In order to apply the method of Jacobi's last multiplier, equation (28) is written as the system of two nonautonomous first-order ordinary differential equations:

$$u' = u_x, \qquad u_x' = -\frac{(2u_x y + 1)u + u_x y}{uy}u_x \tag{31}$$

with u_x a new obvious dependent variable. Of course, we have trivially to construct the first prolongations of the generators (30) in order for each of them to generate a Lie symmetry of system (31). For example the first to the prolongation of X_1 is

$$\underset{1}{X_1} = \tfrac{1}{4}\,\mathrm{e}^{2u}(2u-1)y\partial_y - \tfrac{1}{4}\,\mathrm{e}^{2u}u_x(4uu_x y + 2u - 1)\partial_{u_x}.$$

[3] Note that we have changed the original notation.

The Jacobi's last multiplier of system (31) cannot be constant, in contrast to the Kepler problem, but has to satisfy the following equation:

$$\frac{\mathrm{d}\log(M)}{\mathrm{d}y} = \frac{4uu_x y + 2u_x y + u}{yu}. \tag{32}$$

Note that system (31) is not autonomous which means that the corresponding matrices as given in (13) have three columns and consequently there are 28 possible determinants to be calculated. By way of illustration the matrix that we obtain with the choice of $\underset{1}{X}_1$ and $\underset{1}{X}_3$ is

$$C_{13} = \begin{bmatrix} 1 & u_x & -\dfrac{(2u_x y + 1)u + u_x y}{uy} u_x \\ \dfrac{1}{4}\,\mathrm{e}^{2u}(2u-1)y & 0 & -\dfrac{1}{4}\,\mathrm{e}^{2u} u_x (4uu_x y + 2u - 1) \\ 0 & \dfrac{2u-1}{u} & \dfrac{u_x}{u^2} \end{bmatrix}. \tag{33}$$

First integrals of system (31) are then obviously obtained by taking any ratio of two determinants which are not null. Using Maple 7 it is easy to find that the determinants which are different from zero are the following[4]:

$$\begin{aligned}
\Delta_{14} &= -\frac{u_x}{4u}\,\mathrm{e}^{2u}(-2u + 1 + 4u_x yu\log(y))^2 \\
\Delta_{15} &= u_x^2 y \\
\Delta_{16} &= \frac{u_x}{4u}(-2u + 1 + 4u_x yu\log(y)) \\
\Delta_{17} &= -y^2 u_x^3 u\,\mathrm{e}^{2u} = -u_x yu\,\mathrm{e}^{2u}\Delta_{15} \\
\Delta_{18} &= -\frac{1}{4} yu_x^2\,\mathrm{e}^{2u}(-2u + 1 + 4u_x yu\log(y)) = -u_x yu\,\mathrm{e}^{2u}\Delta_{16} \\
\Delta_{24} &= -\frac{\mathrm{e}^{2u}}{16u^2 y}(-2u + 1 + 4u_x yu\log(y))^3 \\
\Delta_{25} &= \frac{u_x}{4u}(-2u + 1 + 4u_x yu\log(y)) = \Delta_{16} \\
\Delta_{26} &= \frac{1}{16u^2 y}(-2u + 1 + 4u_x yu\log(y))^2 = -\frac{1}{4u_x yu\,\mathrm{e}^{2u}}\Delta_{14} \\
\Delta_{27} &= -\frac{1}{4} yu_x^2\,\mathrm{e}^{2u}(-2u + 1 + 4u_x yu\log(y)) = \Delta_{18} = -u_x yu\,\mathrm{e}^{2u}\Delta_{16} \\
\Delta_{28} &= -\frac{u_x}{16u}\,\mathrm{e}^{2u}(-2u + 1 + 4u_x yu\log(y))^2 = -u_x yu\,\mathrm{e}^{2u}\Delta_{26} = \frac{1}{4}\Delta_{14} \\
\Delta_{34} &= \frac{1}{u^2 y}(-2u + 1 + 4u_x yu\log(y))^2 = 16\Delta_{26} = -\frac{4}{u_x yu\,\mathrm{e}^{2u}}\Delta_{14} \\
\Delta_{35} &= -4\frac{u_x}{u\,\mathrm{e}^{2u}} = -\frac{4}{u_x yu\,\mathrm{e}^{2u}}\Delta_{15} \\
\Delta_{36} &= -\frac{1}{u^2 y\,\mathrm{e}^{2u}}(-2u + 1 + 4u_x yu\log(y)) = -\frac{4}{u_x yu\,\mathrm{e}^{2u}}\Delta_{16} \\
\Delta_{37} &= 4u_x^2 y = 4\Delta_{15} \\
\Delta_{38} &= \frac{u_x}{u}(-2u + 1 + 4u_x yu\log(y)) = 4\Delta_{16}
\end{aligned}$$

[4] We use the symbolism Δ_{ij} to mean the determinant of the matrix which has $\underset{1}{X}_i$ and $\underset{1}{X}_j$ in the second and third rows, respectively.

$$\Delta_{45} = \frac{1}{u^2 y\, e^{2u}}(-2u + 1 + 4u_x yu \log(y)) = -\Delta_{36} = \frac{4}{u_x yu\, e^{2u}}\Delta_{16}$$
$$\Delta_{47} = -\frac{u_x}{u}(-2u + 1 + 4u_x yu \log(y)) = -\Delta_{38} = -4\Delta_{16}$$
$$\Delta_{56} = \frac{1}{u^2 y\, e^{4u}} = \frac{1}{(u_x yu\, e^{2u})^2}\Delta_{15}$$
$$\Delta_{58} = -\frac{u_x}{u\, e^{2u}} = \frac{1}{4}\Delta_{35} = -\frac{1}{u_x yu\, e^{2u}}\Delta_{15}$$
$$\Delta_{67} = \frac{u_x}{u\, e^{2u}} = -\frac{1}{4}\Delta_{35} = \frac{1}{u_x yu\, e^{2u}}\Delta_{15}$$
$$\Delta_{78} = u_x^2 y = \Delta_{15}.$$

As a consequence the following are first integrals[5] of system (31) and consequently of equation (28):

$$I_1 = \frac{\Delta_{14}}{\Delta_{15}} = -\frac{e^{2u}}{4u_x yu}(-2u + 1 + 4u_x yu \log(y))^2$$
$$I_2 = \frac{\Delta_{14}}{\Delta_{16}} = -e^{2u}(-2u + 1 + 4u_x yu \log(y))$$
$$I_3 = \frac{\Delta_{14}}{\Delta_{17}} = -u_x yu\, e^{2u}\frac{\Delta_{14}}{\Delta_{15}} = -u_x yu\, e^{2u} I_1 \Longrightarrow$$
$$I_4 = u_x yu\, e^{2u}$$
$$I_5 = \frac{\Delta_{14}}{\Delta_{18}} = -u_x yu\, e^{2u}\frac{\Delta_{14}}{\Delta_{16}} = -I_4 I_2$$
$$I_6 = \frac{\Delta_{14}}{\Delta_{24}} = \frac{4uu_x y}{-2u + 1 + 4u_x yu \log(y)}$$
$$I_7 = \frac{\Delta_{15}}{\Delta_{16}} = \frac{4uu_x y}{-2u + 1 + 4u_x yu \log(y)} = I_6$$
$$I_8 = \frac{\Delta_{15}}{\Delta_{24}} = -\frac{16u^2 u_x^2 y^2\, e^{-2u}}{(-2u + 1 + 4u_x yu \log(y))^3}$$
$$I_9 = \frac{\Delta_{16}}{\Delta_{24}} = -\frac{4uu_x y\, e^{-2u}}{(-2u + 1 + 4u_x yu \log(y))^2}.$$

Naturally we have omitted all the other possible combinations (210 in total) which yield the same first integrals as above or a constant. Finally we note that the first integral I_4 corresponds to the first integral I, i.e. (29), which was determined in [8].

5. Final remarks

We have seen above how the symmetries of the Kepler problem enable us to use the method of Jacobi's last multiplier in order to compute the first integrals of the Kepler problem. We can then infer that other systems of differential equations arising in various branches of physics and other natural sciences possess conservation laws which can be found by the interaction between the Jacobi last multiplier and their symmetries as exemplified within this paper. We have also shown that in order to obtain first integrals one needs only to perform algebraic

[5] We recall that (28) possesses just two functionally independent first integrals and naturally the listed integrals are functions of two variables representing the two preferred integrals.

operations, i.e. evaluate the determinant of a certain matrix, once the symmetries are known. This is a more general method than using Noether's theorem: in fact it can be used even if the systems of differential equations do not come from a variational principle.

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